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On Field's Nominalization of Physical Theories

Abstract. In his book *Science Without Numbers* Hartry Field argues that we can "nominalize" our physical theories, that is we can reformulate them in such a way that (1) the new version preserves the attractivity of the theory, and (2) the nominalized theory does not contain quantification over mathematical entities. I reconsider Field's nominalization procedure for a toy physical theory formulated in a first order language, in order to make a clear distinction between the following three steps: (1) the physical theory in terms of empirical observations; (2) the standard physical theory, which contains quantification over mathematical entities, as usual; (3) the nominalized version of the theory without any reference to mathematical entities. Having Field's nominalization procedure reconstructed, it will be clear that from a formalist point of view there is no difference between the original and the nominalized versions of the theory. It is because the only difference would come from the different "meanings" of the variables over which the quantification is running. The formalist philosophy of mathematics, however, denies that the variables have meanings at all. Finally, I consider further arguments for and against the indispensability of mathematical objects.

1 INTRODUCTION

One of the most important questions in the philosophy of mathematics is the ontological status of mathematical entities. In the late 1970s, Quine and Putnam suggested an argument for the existence of mathematical entities. The argument is based on the idea that mathematics is not only applicable but in fact *indispensable* for the empirical sciences; and if that is the case, then mathematical entities are as indispensable for our best ontological picture of the world as electrons and other physical entities, to the existence of which physicists are committed.

In its most explicit form the argument reads as follows:

Indispensability argument

(P1) We ought to have ontological commitment to all and only the entities that are indispensable to our best scientific theories.

(P2) Mathematical entities are indispensable to our best scientific theories.

(C) We ought to have ontological commitment to mathematical entities.

The argument has attracted a great deal of attention. Many Platonists regard it as the best available argument for the existence of mathematical entities. The opponents of the argument object, first of all, to the first premise; while the second premise is considered uncontroversial (Colyvan 2004).

What does the first premise exactly mean? First of all, it definitely presupposes a kind of naturalism. For naturalism claims that we have to have ontological commitment to all and only the entities that exist according to our best scientific theories. According to Quine—see (Quine 1961), (Quine 1981)—if the language of the scientific theory quantifies over some entities which are, at the same time, indispensable, then we ought to have ontological commitment to those entities. It is therefore necessary to clarify the proper meaning of "indispensability".

It would be quite straightforward to interpret dispensability as eliminability. An entity is eliminable from a theory if there is another theory which is empirically equivalent to the original one, but does not quantify over the entity in question. In this case, however, every non-observable entity would be dispensable, due to the well-known Craig theorem; see (Craig 1953). In other words, we would have to reject the existence of anything but the directly observable entities. In order to avoid such a radical conclusion, many suggest, we need to prescribe some further requirements for the new theory. These requirements are usually the following: clarity, simplicity, unificatory power, generality and fecundity ((Burgess 1983), (Maddy 2005)). These requirements altogether are called *attractivity*. So, an entity is dispensable if it is eliminable and the theory we obtain by its elimination remains attractive. In any event, the notion of attracivity is quite ambiguous; and it is hard to believe that the most fundamental ontological questions depend on such unclear and sociologically relative notions.

On the other hand, the second premise was considered as an evident one. Hartry Field was the first who claimed that the second premise is, in fact, false; mathematical entities are not indispensable to our best scientific theories. Field adopted Quine's linguistic criterion that a scientific theory asserts the existence of an entity by quantifying over the entity in question. He also accepted the attractivity requirements. But he showed that a physical theory can always be "nominalized", by which he means that it can be reconstructed such that (1) the new theory does not contain quantification over mathematical entities, however, (2) remains attractive.

2 THE NOMINALIZATION PROCEDURE

In his *Science Without Numbers*(Field 1980) Field showed, as an example, how a fragment of Newtonian gravitational theory can be nominalized. In what follows, I will reconstruct Field's nominalization procedure in the case of an even simpler "toy" physical theory. My purpose is not only to demonstrate the nominalization steps on a perhaps more clear-cut example, but also to lay the emphasis on different points. First, within a physical theory, we will make a clear separation of the formal system and the semantics. Second, we will keep it clear that the equivalence of physical theories is understood as *empirical* equivalence (Fig. 1).

The nominalization procedure consists of the following three steps:

(1) We have a body of physical facts, in terms of empirical observations.
(2) We have the usual platonistic physical theory describing the observable phenomena in question—containing quantification over mathematical entities. The platonistic theory will consist of a formal system L, and a semantics S.
(3) We construct a new theory which is capable of describing the same phenomena, but without quantification over mathematical entities. The nominalized theory will consist of a formal system L', and a semantics S'. We will show the equivalence of these theories on the level of observable phenomena.

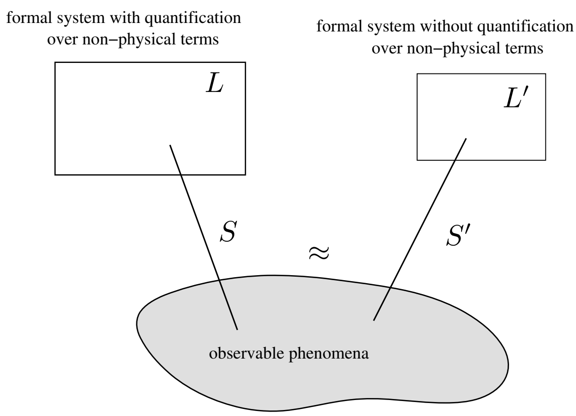

Figure 1. The platonistic physical theory and the nominalized physical theory should be equivalent on the empirical level.

3 THE "TOY" PHYSICAL THEORY

The "toy" physical theory is about a few empirically observable regularities related with the spatial relations of properties of the material points/molecules of a sheet of paper. Thus the only physical entities are the molecules of the paper and the only measuring equipment will be a scale-free ruler. We will examine—empirically—the following two properties of the molecules:

Betweenness We say that molecule γ is between molecules α and β if whenever the ruler fits to α and β then it also fits to γ and the mark on the ruler corresponding to γ falls between the marks corresponding to α and β.

Congruence We will say that a pair of molecules α, β is congruent to a pair of molecules γ, δ if whenever we mark the ruler at α, β, the same marks will also fit to γ, δ.

With our scale-free ruler we can observe the following empirical facts about the molecules of the paper:

(E1) If molecules α and β are congruent to molecules γ and δ, and γ and δ are congruent to molecules ε and ζ, then α and β are congruent to ε and ζ.

(E2) If we consider three molecules fitting to the ruler, then there is exactly one that lies between the other two.

4 THE USUAL PLATONISTIC PHYSICAL THEORY OF THE PAPER

We present a "platonistic" physical theory (L, S) which describes the empirical facts of the paper. The formal system of the physical theory will be $L = (\mathbb{R}^2, \Gamma, \Lambda)$, where

$\Gamma(a, b, c)$ is a relation between three points of $\mathbb{R}^2$ (six real numbers):

$$\Gamma(a,b,c) \iff \sqrt{(a_1-b_1)^2+(a_2-b_2)^2}+\sqrt{(c_1-b_1)^2+(c_2-b_2)^2}$$

$$= \sqrt{(a_1-c_1)^2+(a_2-c_2)^2}$$

$\Lambda(a, b, c, d)$ is a relation between four points of $\mathbb{R}^2$ (eight real numbers):

$\Lambda(a, b, c, d) \iff$

$$\sqrt{(a_1-b_1)^2+(a_2-b_2)^2} = \sqrt{(c_1-d_1)^2+(c_2-d_2)^2}$$

The semantics S is defined as follows: First, to every molecule we assign an element of $\mathbb{R}^2$: α corresponds to $a = (a_1, a_2) \in \mathbb{R}^2$, β to $b = (b_1, b_2) \in \mathbb{R}^2$, and so on. Second, relation Γ corresponds to the *Betweenness* and Λ corresponds to the *Congruence* of the molecules. All this representation is carefully constructed so that the physical theory (L, S), that is, the formal system $(\mathbb{R}^2, \Gamma, \Lambda)$ with the above semantics provides a proper description of our empirical knowledge about the paper. It means that if $\Gamma(a,b,c)$ is true for $a,b,c \in \mathbb{R}^2$ then it is true for the corresponding molecules α, β and γ that molecule β is between α and γ. Similarly, if $\Lambda(a,b,c,d)$ is true for $a,b,c,d \in \mathbb{R}^2$ then it is true that molecules α, β are congruent to molecules γ, δ.

For example, empirical facts (E1) and (E2) are obviously represented in the theory (L,S). Moreover, (L,S) has predictive power. For instance, in $(\mathbb{R}^2, \Gamma, \Lambda)$ we can prove the following theorem (Fig. 2):

Theorem 1.

$$\begin{aligned}
&\forall a \forall b \forall g \forall d \forall e \forall z \exists o\ \Gamma(a,d,b) \wedge \Gamma(b,e,g) \wedge \Gamma(g,z,a) \\
&\wedge \Lambda(a,d,d,b) \wedge \Lambda(b,e,e,g) \wedge \Lambda(g,z,z,a) \\
&\rightarrow \Gamma(a,o,e) \wedge \Gamma(b,o,z) \wedge \Gamma(g,o,d)
\end{aligned}$$

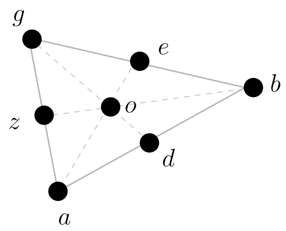

Figure 2. The centroid of a triangle always exists.

With the above semantics, we arrive at the following *hypothesis* about the molecules of the paper (Fig. 3):

Hypothesis If molecules $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ satisfy that δ is between α and β, ε is between β and γ, and ζ is between γ and α, furthermore, α, δ are congruent to δ, β, and β, ε are congruent to ε, γ, and γ, ζ are congruent to ζ, α, then we can always find a molecule ω such that it is between α and ε, and it is between β and ζ and it is between γ and δ.

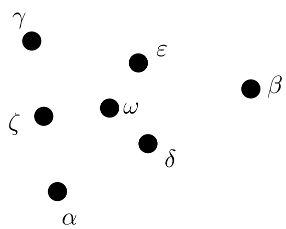

Figure 3. According to the semantics of the theory, **Theorem 1.** (Fig. 2) is a statement about the molecules of the paper.

This hypothesis can be verified empirically by means of the scale-free ruler; and we will find that the hypothesis is true.

In Field's terminology, (L, S) is a *platonistic* theory: It contains quantification over mathematical entities, namely, over real numbers, since Γ and Λ are relations between real numbers.

5 THE NOMINALIZED THEORY

Now we will construct another physical theory which can equally well describe the same observable phenomena but without quantification over mathematical entities. This will consist of another formal system L' with another semantics S'.

The formal language L' will be a first order formal system. L' will contain individuum variables $A, B, C, \ldots$ and a three-argument predicate symbol Bet and a four-argument predicate symbol $Cong$. Beyond the logical axioms of PC(=) (predicate calculus with identity), we will need the following "physical" axioms:[1]

T1: $\forall A \forall B\ Cong(A, B, B, A)$
T2: $\forall A \forall B \forall C\ Cong(A, B, C, C) \rightarrow A = B$
T3: $\forall A \forall B \forall C \forall D \forall E \forall F\ Cong(A, B, C, D) \wedge Cong(C, D, E, F)$
$\rightarrow Cong(A, B, E, F)$
T4: $\forall A \forall B\ Bet(A, B, A) \rightarrow A = B$
T5: $\forall A \forall B \forall C \forall D \forall E\ Bet(A, D, C) \wedge Bet(B, E, C))$
$\rightarrow \exists F\ (Bet(D, F, B) \wedge Bet(E, F, A)$
T6: $\exists E \forall A \forall B\ \phi(A) \wedge \psi(B) \rightarrow Bet(E, A, B)$
$\rightarrow \exists F \forall A \forall B\ \phi(A) \wedge \psi(B) \rightarrow Bet(A, F, B)$
where ϕ and ψ are two arbitrary formulas of the language, containing no free instances either E or F. Let there also be no free instances of A in $\psi(B)$ or of B in $\phi(A)$.
T7: $\exists A \exists B \exists C\ \neg Bet(A, B, C) \wedge \neg Bet(B, C, A) \wedge \neg Bet(C, A, B)$
T8: $\forall A \forall B \forall C \forall D \forall E\ Cong(A, D, A, E) \wedge Cong(B, D, B, E)$
$\wedge\ Cong(C, D, C, E) \wedge \neg D = E$
$\rightarrow Bet(A, B, C) \vee Bet(B, C, A) \vee Bet(C, A, B)$
T9: $\forall A \forall B \forall C \forall D \forall E\ Bet(A, D, E) \wedge Bet(B, D, C) \wedge \neg A = D$
$\rightarrow \exists F \exists G\ Bet(A, B, F) \wedge Bet(A, C, G) \wedge Bet(G, E, F)$
T10: $\forall A \forall B \forall C \forall D \forall E \forall F \forall G \forall H\ \neg A = B \wedge Bet(A, B, C) \wedge Bet(E, F, G)$
$\wedge\ Cong(A, B, E, F) \wedge Cong(B, C, F, G) \wedge Cong(A, D, E, H)$
$\wedge\ Cong(B, D, F, H) \rightarrow Cong(C, D, G, H)$
T11: $\forall A \forall B \forall C \forall D \exists E\ Bet(D, A, E) \wedge Cong(A, E, B, C)$

[1]The reader may recognize that these are nothing but the well known Tarski–Givant (1999) axioms of Euclidean geometry. But it must be emphasized that this fact is irrelevant.

The S' semantics of the theory is defined as follows. The individuum variables $A, B, C, \ldots$ will refer to the molecules of the paper. The predicate symbol Bet corresponds to the *Betweenness*, and the predicate symbol $Cong$ corresponds to the *Congruence*.

The physical theory (L', S') with the above semantics provides a proper description of our empirical knowledge about the paper. For example, empirical facts (E1) and (E2) are obviously represented by theorems in (L', S'). This theory equally well describes our empirical knowledge about the paper. It also has the same predictive power. For instance, in L' we can prove the following theorem Fig. 4:

Theorem 1′.

$$\begin{aligned}
&\forall A \forall B \forall G \forall D \forall E \forall Z \exists O \; Bet(A, D, B) \\
&\wedge Bet(B, E, G) \wedge Bet(G, Z, A) \wedge Cong(A, D, D, B) \\
&\wedge Cong(B, E, E, G) \wedge Cong(G, Z, Z, A) \\
&\rightarrow Bet(A, O, E) \wedge Bet(B, O, Z) \wedge Bet(G, O, D)
\end{aligned}$$

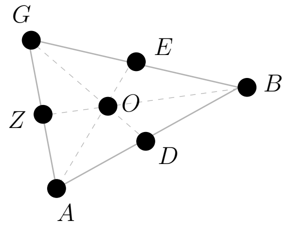

Figure 4. The centroid of a triangle always exists.

With the above semantics, this leads us to the following hypothesis about the molecules of the paper (Fig. 5):

Hypothesis If molecules $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ satisfy that δ is between α and β, ε is between β and γ, and ζ is between γ and α, furthermore, α, δ are congruent to δ, β, and β, ε are congruent to ε, γ, and γ, ζ are congruent to ζ, α, then we can always find a molecule ω such that it is between α and ε, and it is between β and ζ and it is between γ and δ.

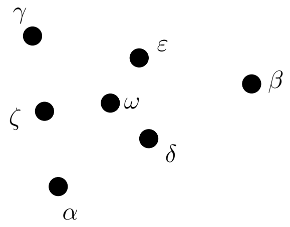

Figure 5. According to the semantics of the theory, Theorem 1′. (Fig. 4) is a statement about the molecules of the paper.

This hypothesis can be verified empirically by means of the scale-free ruler; and we will find that the hypothesis is true.

6 CONCLUDING REMARKS

As we can see, quantification over the mathematical entities can be eliminated from a physical theory. For example, in this "toy" physical theory we eliminated quantification over the points of $\mathbb{R}^2$ and quantification over the real numbers. This does not mean, however, that we have really purified our physical theory from Platonic objects: Although we eliminated *quantification* over mathematical entities, we did not eliminate the *mathematical structures* themselves. We still need the structure defined by axioms T1–T11. It thus seems unavoidable to draw the conclusion that we ought to have ontological commitment to formal systems as abstract entities; and this is sufficient for the structuralist version of Platonism (Shapiro 1997).

It must be noted that both the Quine–Putnam argument and Field's criticism are based on the tacit assumption that the terms and statements of mathematics have *meanings*, and the only question is the ontological status of the entities that mathematics refers to. According to the formalist philosophy of mathematics, however, this assumption is unacceptable, *ab ovo*.

As we have seen in our example, both the *platonistic* and the *nominalized* versions of the physical theory have the same structure: a *meaningless formal system* + a *partial semantics* pointing only to physical, moreover, observable things. From this point of view, it does not matter whether or not the formal system in question contains quantification over certain variables. Formal systems are obviously indispensable from both platonistic and nominalized physical theories, in spite of the fact that they are meaningless. The only question is: What is the ontological status of formal systems? And still, one can answer this question from a structuralist–Platonist position, drawing support from the Quine–Putnam indispensability argument. Or, one can consider an entirely different account for formal systems, which completely intact from the indispensability argument (see for example (Szabó 2003) for a physicalist ontology of formal systems).*

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