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Fitch's paradox and Labeled Natural Deduction System

Abstract. This paper introduces a relatively novel system of representing modal logic in a form of natural deduction. It then expands it to accommodate the epistemic operator and applies it to generate a more precise formulation of Fitch's paradox of knowability. Finally, an illustration of the paradox's pertinence to contemporary philosophical debate is laid out.

1 INTRODUCTION

The purpose of this paper is to provide the means for presenting Fitch's paradox, a philosophical argument requiring multiple modalities, within a purely formal deduction system. A labeled natural deduction system for modal logic offered by David Basin, Sean Matthews and Luca Vigano, provides the basis which is then expanded to accommodate an epistemic operator "know." An advantage of this system: anyone familiar with first-order natural deduction is provided with the means to formulate a useful and fruitful philosophical argument in a more precise manner at no added complexity.

The remainder of this section will lay out some desirable properties of any natural deduction system that we will naturally strive to meet in this paper. The second section introduces the labeled natural deduction system of Basin *et al.* and expands on it to allow us to formulate Fitch's paradox. Note that, while the authors use the "Gentzen-style" form of representing natural deduction, due to the relative length of the argument and the number of assumptions needed, for ease of presentation, the form used here is the "Suppes–Lemmon style." The third section presents Fitch's paradox first in an informal, and then in a formal manner. Finally, the fourth section provides the summary.

1.1 Natural deduction systems

Although natural deduction was first developed 1934, it is partially based on a proposal put forth in 1926 by Jan Lukasiewicz, who called for a system that can yield the same theorems as the axiomatic systems of the time, but which would follow more closely the actual practice of constructing a proof[1]. This system, reflecting the "natural" way humans reason, would follow where "arbitrary assumptions" lead and how long they stay in effect. This desirable property is something to keep in mind while presenting a natural deduction system.

In their widely used logic handbook *Language, Proof and Logic* Jon Barwise and John Etchemendy state that: "… [natural deduction] systems are intended to be models of the valid principles of reasoning used in informal proofs."[2] This is precisely the purpose of the natural deduction system they present.

2 LABELED NATURAL DEDUCTION SYSTEM FOR MODAL LOGIC

This section introduces a labeled natural deduction system developed by Basin *et al.* Following the ideas of Dov Gabbay, this system provides a framework for capturing a large number of non-classical logics[3]. The focus here will be on modal logic. The peculiarity of the system is that it introduces a set of labels W, $W = \{x_0, x_1, \dots, x_n, \dots, y_0, y_1, \dots, y_n, \dots\}$. These labels can be thought of as representing worlds in aKripke model. The language of the labeled natural deduction system (henceforth: LNDS) differs from the standard, and widely familiar, (propositional) modal logic language precisely with regard to W.

2.1 The language of LNDS

The language of LNDS comprises two types of formulas, labeled and relational well-formed formulas. The latter concern the relations of labels, and correspond to the properties of the accessibility relation R in a Kripke model, whereas the former are merely modal propositional wel formed formulas expanded with a label; we will define these first, and proceed from there. The (inductive) definition of a modal *wff* should be familiar:

[1] (Pelletier, F., 2000).
[2] (Barwise, J., Etchemendy, J., 2003).
[3] (Basin *& al.* 1998).

Definition 2.1: Modal *wff*
1 Propositional letters P, Q and R are well formed formulae (*wff*).
2 If P is a *wff*, then $\neg P$ is a *wff*.
3 If P is a *wff* and Q is a *wff*, then $(P \wedge Q)$ is a *wff*.
4 If P is a *wff* and Q is a *wff*, then $(P \vee Q)$ is a *wff*.
5 If P is a *wff* and Q is a *wff*, then $(P \rightarrow Q)$ is a *wf.f*
6 If P is a *wff* Q is a *wff*, then $(P \leftrightarrow Q)$ is a *wff*.
7 If P is a *wff*, then $\Box P$ is a *wff*.
8 If P is a *wff*, then $\Diamond P$ is a *wff*.
9 Nothing else is a *wff*.
Now we have all the ingredients necessary to define a *labeled well-formed formula*:

Definition 2.2: Labeled well-formed formula (*lwff*)
Let P be a modal *wff* (Def. 2.1), let W be a set of labels $W = \{x_0, x_1, \ldots, x_n, \ldots ,y_0, y_1, \ldots, y_n, \ldots\}$, and let x be a member of such a set, $x \in W$. Then $x{:}P$ is a *lwff* which can be understood as meaning "P is the case in (a possible world) x."
As noted earlier, a *relational wff* is concerned with a relation of two labels:

Definition 2.3: Relational well formed formula (*rwff*)
Let x and y be members of a set of labels W (as above). Then xRy is a *rwff* which can be understood as "y is accessible to x."

2.2 Rules of inference

In this section we will explore the rules of inference in LNDS. The rules of inference for truth-functional connectives should be readily recognizable to anyone familiar with natural deduction—the only novelty here being that each line is expanded with (one and the same) label. The rule for negation introduction deviates from this pattern, and will be discussed separately. Afterwards, rules of inference for modal operators will be covered, along with examples to illustrate them. The mode of presentation is "Suppes–Lemmon" style—the central column contains the enumerated steps of the proof, assumptions or derived formulas. The column on the right contains the "justification" of a step—a rule of inference used, or "P" if it is an assumption ("P*" denotes an additional assumption that needs to be discharged, and the column on the left contains a set of undischarged assumptions the step "relies" on (assumptions "rely" on themselves). In a general form laid out here, the letters m, n, i, j, … signify numbers, Greek letters Γ and Δ signify sets of assumptions, and letters A, B, … signify *wff*s.

2.2.1 Truth-functional connectives

Below are the rules for conditional and conjunction, which behave in a familiar way, the only difference being that each *wff* is expanded into a *labeled wff*, using the same label in each instance.

$\rightarrow$ *Intro*

$\{m\}$	m	$x{:}A$	P^*
		. . .	
$\Gamma \cup \{m\}$	i	$x{:}B$	
Γ	j	$x{:}\ A \rightarrow B$	$\rightarrow I{:}\ m,i$

$\rightarrow$ *Elim*

Γ	m	$x{:}\ A \rightarrow B$	
Δ	i	$x{:}A$	
$\Gamma \cup \Delta$	j	$x{:}B$	$\rightarrow E{:}\ m,i$

$\wedge$ *Intro*

Γ	m	$x{:}A$	
Δ	i	$x{:}B$	
$\Gamma \cup \Delta$	j	$x{:}\ A \wedge B$	
		or	$\wedge I{:}\ m,i$
$\Gamma \cup \Delta$	j	$x{:}\ A \wedge B$	$\wedge I{:}\ m,i$

$\wedge$ *Elim*

Γ	m	$x{:}\ A \wedge B$	
Γ	i	$x{:}A$	$\wedge E{:}\ m$
	or		
Γ	i	$x{:}B$	$\wedge E{:}\ m$

$\neg$ *Intro*

$\{m\}$	m	$x{:}A$	P^*
		. . .	
$\Gamma \cup \{m\}$	i	$y{:}\bot$	
Γ	j	$x{:}\ \neg A$	$\neg I{:}\ m,i$

Or, alternatively

$\neg$ *Intro*

$\{m\}$	m	$x{:}\neg A$	P^*
		. . .	
$\Gamma \cup \{m\}$	i	$y{:}\bot$	
Γ	j	$x{:}A$	$\neg I{:}\ m,i$

Note that not all the lines here contain the same label (the label used in the line (*i*) is "*y*"*)*. An impossible result in one world (i.e. under one label) can "transfer" to another world—a fact that Basin *et al.* call a "global falsum." This is elaborated in Section 2.4.

$\bot$ *Intro*

Γ	m	$x{:}A$	
Δ	i	$x{:}\neg A$	
$\Gamma \cup \Delta$	j	$y{:}\bot$	$\bot I$: m,i

$\bot$ *Elim*

Γ	n	$x{:}\bot$	
Γ	i	$x{:}A$	$\bot E$: n

2.2.2 Modal operators

The novelty of this approach consists in the introduction of natural deduction rules for the modal operators "□" ("necessarily") and "◊" ("possibly"). Note that these rules of inference are analogous to the rules for universal and existential quantifiers, respectively (with an "arbitrary label" replacing an "arbitrary name").

□ *Intro*

$\{m\}$	m	xRy	P*
		. . .	
Γ	i	$y{:}A$	
$\Gamma - \{m\}$	j	$x{:}\Box A$	$\Box I$: m,i

Note: y is a new label, such that $x \neq y$, and not appearing in any of the suppositions in Γ, except perhaps $\{m\}$.

□ *Elim*

Γ	m	$x{:}\Box A$	
Δ	i	xRy	
$\Gamma \cup \Delta$	j	$y{:}A$	$\Box E$: m,i

Example 2.1: See the Appendix.

◊ *Intro*			
Γ	*m*	*x*:*A*	
Δ	*i*	*xRy*	
Γ ∪ Δ	*j*	*x*: ◊ *A*	◊ *I*: *m*,*n*
◊ *Elim*			
Γ	*m*	*x*:*A*	*P**
{*i*}	*i*	*xRy*	*P**
{*j*}	*j*	*x*: ◊ *A*	◊ *I*: *m*,*n*
		. . .	
Δ ∪ {*i*} ∪ {j}	*k*	*z*:*B*	
Γ ∪ Δ	*l*	*z*:*B*	◊ *E*: *m*,*i*,*j*,*k*

Note: y is a new label, such that $y \neq x$ and $y \neq z$, which does not appear in any of the suppositions from Γ and Δ.

Example 2.2: See the Appendix.

2.3 Familiar axioms

As noted, it is expected of a natural deduction system that it provide the same results as an axiomatic theory. Therefore, what follows are proofs of two well-known modal axioms—the rule of necessitation, which states that every theorem is necessary, and axiom K, which demonstrates how the necessity operator "□" is distributed over conditionals.

Proof 2.1: Rule of Necessitation (*RN*)
Let *x:A* be a theorem, and *x* an arbitrary label. Proof for *x:* □ *A* will proceed as follows:

{*m*}	*m*	*xRy*	*P**
		the proof of a theorem where each occurrence of the label x is substituted for the label y	
{}	*n*	*y*:*A*	*from the preceding proof*
{}	*j*	*x*: □ *A*	□ *I*: *m*,*n*

Proof 2.2: Axiom K

{1}	1	x: $\Box(A \to B)$	P^*
{2}	2	x: $\Box A$	P^*
{3}	3	xRy	P^*
{1,3}	4	y: $A \to B$	$\Box E$: 1,3
{2,3}	5	$y{:}A$	$\Box E$: 2,3
{1,2,3}	6	$y{:}B$	$\to E$: 4,5
{1,2}	7	x: $\Box B$	$\Box I$: 3,6
{1}	8	x: $\Box A \to \Box B$	$\to I$: 2,7
{}	9	x: $\Box(A \to B) \to (\Box A \to \Box B)$	$\to I$: 1,8

2.4 Global falsum

One consequence of the negation introduction rule is the rule called "global falsum": $\Gamma \vdash_{x:\bot} \Rightarrow \Gamma \vdash_{x:\bot}$.[4]

Proof 2.3: Global falsum (*gf*)
Suppose that (1) $\Gamma \vdash x{:}\bot$. Then Γ, $y{:}P \vdash x{:}\bot$ (adding a premise does not alter the validity of a valid argument). It follows by $\neg I$ that (2) $\Gamma \vdash y{:}\neg P$. But in the same way, from (1) we can derive Γ, y: $\neg P \vdash x{:}\bot$, and another application of $\neg I$ gives (3) $\Gamma \vdash y{:}P$. Applying $\bot I$ to (2) and (3) yields $\Gamma \vdash y{:}\bot$.
Since x and y represent arbitrary labels, it is obvious that falsum can "travel" freely between labels. The reason for the inclusion of the rule *global falsum* is that it allows a desirable result—interchangeability of W and $\neg \Diamond \neg$.

Global falsum

Γ	1	$x{:}\bot$	*gf*:*m*
Γ	2	$x{:}\bot$	

The following two proofs demonstrate how this inference rule allows for the derivation of that desirable result.

[4] (Basin *& al.* 1998).

Proof 2.4a

{1}	1	x: □ A	P
{2}	2	x: ◊ ¬ A	P*
{3}	3	y: ¬ A	P*
{4}	4	xRy	P*
{1,4}	5	y:A	□ E: 1,4
{1,3,4}	6	y:⊥	⊥ I: 3,5
{1,3,4}	7	x:⊥	gf: 5
{1,2}	8	x:⊥	◊ E: 2,3,4,6
{1}	9	x: ¬ ◊ ¬ A	¬ I: 2,7

Proof 2.4b

{1}	1	x: ¬ ◊ ¬ A	P
{2}	2	xRy	P*
{3}	3	y: ¬ A	P*
{2,3}	4	x: ◊ ¬ A	◊ I: 2,3
{1,2,3}	5	x:⊥	⊥ I: 1,4
{1,2}	6	y:A	¬ I: 3,5
{1}	7	x: □ A	□ I: 2,6

2.5 Relational rules

Relational rules have the general form $t_1Rs_1 \ . \ . \ . \ t_mRs_m \vdash t_0Rs_0$, where $t_0,t_1,\ldots,t_m,s_0,s_1,\ldots,s_m$ are members of the set of labels W. Relational rules mirror properties of the accessibility relation, and allow us to derive the corresponding axioms. The only rule necessary for the construction of Fitch's paradox is the relational rule of reflexivity, and it is therefore the only one presented here.

Reflexivity

	M		
{}	*I*	xRx	*Rrefl*:

Proof 2.5: axiom T

{1}	1	xRx	*Rrefl*:
{2}	2	$x: \Box A$	P^*
{2}	3	$x{:}A$	$\Box E$: 1,2
{}	4	$x: \Box A \rightarrow A$	$\rightarrow I$: 2,3

It is clear how this is in keeping with the historical requirement posed for natural deduction—to yield the same results as an axiomatic theory.

3 FITCH'S PARADOX

Fitch's paradox, also known as the paradox of knowability, first appeared in Fitch's 1963 article "*A Logical Analysis of Some Value Concepts.*" There, the paradox appears in Theorem 5, which states:

> If there is some true proposition which nobody knows (or has known or will know) to be true, then there is a true proposition which nobody can know to be true.[5]

However, an equivalent claim, which states that if all truths are knowable, then all truths are known, is usually considered when discussing the paradox:

$$\forall p\ (p \rightarrow \Diamond Kp) \vdash \forall p\ (p \rightarrow Kp)$$

3.1 Informal proof of the paradox

The strength of the paradox derives from the fact that it rests on mostly unproblematic principles. They are:

$$(KIT){:}\ Kp \vdash p,$$

which states that knowledge is factive, i.e. knowledge implies truth.

$$(K\ Dist){:}\ K\ (p \wedge q) \vdash Kp \wedge Kq,$$

which states that knowledge is distributed over conjunction, i.e. knowledge of a conjunction implies knowledge of the conjuncts.

[5] (Fitch, F., 1963).

Furthermore, we must rely upon the rule of necessitation (*RN*)—all theorems are necessary. The fourth and final principle states that if p is necessarily false, it is impossible:

$$(\text{P4}): \Box \neg p \vdash \neg \Diamond p$$

Proof 3.1: Fitch's paradox[6]

Now, suppose that every truth is knowable: $\forall p\ (p \to \Diamond K p)$. Suppose also that we are not omniscient, that there is a truth which is not known: $\exists p\ (p \wedge \neg K p)$. Let p be such a truth:

$$(1)\ p \wedge \neg K p$$

Now, since every truth is knowable, so is (1):

$$(2)\ (p \wedge \neg K p) \to \Diamond K (p \wedge \neg K p)$$

Therefore, by *modus ponens*,

$$(3)\ \Diamond K (p \wedge \neg K p)$$

This, however, can be proven to be false. Let us suppose (for *reductio ad absurdum*):

$$(4)\ K (p \wedge \neg K p)$$

It follows by *K Dist* that both conjuncts are known:

$$(5)\ K p \wedge K \neg K p$$

And, applying *KIT* to the second conjunct, we get a contradiction:

$$(6)\ K p \wedge K p$$

That allows us to negate (4):

$$(7)\ \neg K (p \wedge \neg K p)$$

And, since (7) is a theorem, we can apply *RN* to get:

$$(8)\ \Box \neg K (p \wedge \neg K p)$$

Applying the fourth principle, *P4*, we get the opposite of (3):

$$(9)\ \neg \Diamond K (p \wedge \neg K p)$$

[6] (Brogaard, B., Salerno, S., 2008).

Obviouslym this means there is no unknown truth

$$(10)\ \neg \exists p\ (p \wedge \neg K p)$$

Or, in other words, that all truths are known:

$$(11)\ \forall p\ (p \rightarrow K p)$$

So, supposing that all truths are knowable leads us, very convincingly, to the conclusion that all truths are, in fact, known.

3.2 Formal proof of the paradox

Obviously, for the proof of the paradox to be constructed, we need to have rules for the operator *K*. These rules will mirror the inference rules for the necessity operator (using a separate accessibility relation, R_E) and will allow us to derive all the principles needed in the informal proof.

K Intro

$\{m\}$	m	xR_Ey	P^*
		. . .	
$\Gamma \cup \{m\}$	i	$y{:}p$	
Γ	j	$x{:}\ K\,p$	$K\,I$: m,i

K Elim

Γ	m	$x{:}\ K\,p$	
Δ	i	xR_Ey	
$\Gamma \cup \Delta$	j	$y{:}p$	$K\,E$: m,i

Additionally, the relational rule of reflexivity for R_E will be introduced. It insures the *KIT* principle in keeping with the proof of axiom T in *Proof 2.5*.

Proof 3.2: *K Dist*

{1}	1	$x: K(p \wedge q)$	*P*
{2}	2	$xR_E y$	*P**
{1,2}	3	$y: p \wedge q$	*KE*: 1,2
{1,2}	4	$y: p$	$\wedge$ *E*: 3
{1}	5	$x: K p$	*K I*: 2,4
{6}	6	$xR_E z$	*P**
{1,6}	7	$z: p \wedge q$	*KE*: 1,6
{1,6}	8	$z: q$	$\wedge$ *E*: 7
{1}	9	$x: K q$	*KI*: 6,8
{1}	10	$x: K p \wedge K q$	$\wedge$ *I*: 5,9

Proof 3.3: *KIT*

{1}	1	$x: K p$	*P*
{}	2	$xR_E x$	R_E *refl*:
{1}	3	$x: p$	*KE*: 1,2

Obviously, the remaining principles have already been proven—*RN* in the Proof 2.1, and *P4* in the Proof 2.4b, substituting $\neg p$ for *A*.

Proof 3.4: Fitch's paradox in LNDS

The formal version of the proof starts out in the same way—assuming that *p* is an unknown truth (2), but that every truth is knowable, and thus $p \wedge \neg K p$ as well as (1). Again, this leads to the claim that it is possible to know that something is an unknown truth (3). We now set out to prove (in line 23) that this not the case. Note that for the sake of legibility, the words *label* and *world* are used interchangeably.

{1}	1	$x: (p \wedge \neg K p) \rightarrow \Diamond K(p \wedge \neg K p)$	*P*
{2}	2	$x: p \wedge \neg K p$	*P**
{1,2}	3	$x: \Diamond K(p \wedge \neg K p)$	$\rightarrow$ *E*: 1,2

We assume that there is an accessible world *y* in which it is known that *p* is an unknown truth:

{4}	4	$xR_A y$	*P**
{5}	5	$y: K(p \wedge \neg K p)$	*P**

However, in that case $p \wedge \neg K p$ holds in y, and therefore, $\neg Kp$ holds in y. These steps correspond to an application of principles *K Dist* and *KIT*.

{}	6	$yR_E y$	$R_E refl$:
{5}	7	$y: p \wedge \neg K p$	*KE*: 5,6
{5}	8	$y: \neg K p$	$\wedge E$: 7

At the same time, if $p \wedge \neg K p$ is known in y, then $K p$ holds in y.

{9}	9	$yR_E z$	P^*
{5,9}	10	$z: p \wedge \neg K p$	*KE*: 5,9
{5,9}	11	$z: p$	$\wedge E$: 10
{5}	12	$z: K p$	*KI*: 9,11

Lines 8 and 12 are contradictory—they imply that something is an unknown truth can not be known.

{5}	13	$y: \bot$	$\bot I$: 8,12
{}	14	$y: \neg K (p \wedge \neg K p)$	$\neg I$: 5,13

Since y is an arbitrary world, it is necessarily unknowable that something is an unknown truth. This step corresponds to the line (8) of the informal proof.

{}	15	$x: \Box \neg K (p \wedge \neg K p)$	$\Box I$: 4,14

Now we need to perform the transformation from line (9) of the informal proof. To do so, we will assume that the opposite holds:

{16}	16	$x: \Diamond K (p \wedge \neg K p)$	P^*

Of course, if K (p $\wedge \neg K p$) is possible, then there is a world in which it is true:

{17}	17	$y: K (p \wedge \neg K p)$	P^*
{18}	18	$xR_A y$	P^*

But, even in that world, $\neg K (p \wedge \neg K p)$ is true (since it is, according to line 15, necessary). This leads to a contradiction:

{18}	19	$y: \neg K (p \wedge \neg K p)$	$\Box E$: 15,18
{17,18}	20	$y: \bot$	$\bot I$: 17,19

That contradiction transfers back to the original world:

{17,18}	21	$x{:}\bot$	*gf*: 20

And so the assumption in line *16* proves, be false, as we had hoped.

{16}	22	$x{:}\bot$	$\Diamond E$: 16,17,18,21
{}	23	$x{:}\ \neg \Diamond K (p \wedge \neg K p)$	$\neg I$: 16,22

Finally, we have shown that there are no unknown truths:

{1,2}	24	$x{:}\bot$	$\bot I$: 3,23
{1}	25	$x{:}\ \neg (p \wedge \neg K p)$	$\neg I$: 2,24

Of course, since *p* is an arbitrary proposition, it can be shown that all truths are, in fact, known. This transformation is trivial:

{26}	26	$x{:}p$	P^*
{27}	27	$x{:}\ \neg K p$	P^*
{26,27}	28	$x{:}\ p \wedge \neg K p$	$\wedge I$: 26,27
{1,16,27}	29	$x{:}\bot$	$\bot I$: 25,28
{1,26}	30	$x{:}\ K p$	$\neg I$: 27,29
{1}	31	$x{:}\ p \rightarrow K p$	$\rightarrow I$: 26,30

We have thus arrived at the conclusion that if all truths *can* be known, than all truths *are* known—Fitch's paradox of knowability.

3.3 Philosophical implications of the paradox—an illustration

The purpose of this section is to demonstrate that Fitch's paradox is not just of logical significance—it also makes a genuine and insightful philosophical contribution.

Timothy Williamson uses Fitch's paradox in his book *Knowledge and its limits*[7] to demonstrate, predictably, what the limits of our knowledge are. Let us briefly examine how. Williamson labels the thesis that all truths are known as *strong verificationism* (*SVER*):

$$SVER{:}\ \forall p\ (p \rightarrow K p)$$

[7] (Williamson, T., 2000).

This is, as Williamson puts it, an “insane sounding thesis” (p. 271). The more plausible sounding thesis that all truths are knowable Williamson calls *weak verificationism* (*WVER*):

$$WVER\text{: } \forall p\ (p \rightarrow \Diamond\, K\, p)$$

Obviously, the stronger thesis implies the weaker one, since all that is known can be known. But in order to demonstrate some limits to our knowledge, Williamson uses Fitch’s paradox to demonstrate that the converse also holds—*WVER* implies *SWER*. They are therefore equivalent, and any objection to the insane-sounding thesis will apply to the more plausible formulation as well.

4. SUMMARY

We put forth two desirable qualities of natural deduction systems at the beginning of this paper. The first—that it provide the same theorems as an axiomatic theory by way of making and following arbitrary assumptions—has clearly been met: we have derived all the axioms needed to prove one famous theorem. Regarding the second property, we have used the system to model the principles of reasoning present in the informal proof. So this was, in a manner of speaking, a textbook example of what natural deduction is supposed to do. Moreover, the formal principles come with no added complexity for someone familiar with first-order natural deduction, yet they are able to contribute to a fruitful philosophical debate.

APPENDIX

Example 2.1

{1}	1	$x\text{: } \Box\,(A \wedge B)$	P
{2}	2	xRy	P^*
{1,2}	3	$y\text{: } A \wedge B$	$\Box\,E$: 1,2
{1,2}	4	$y\text{:}A$	$\wedge\,E$: 3
{1}	5	$x\text{: } \Box\,A$	$\Box\,I$: 2,4
{6}	6	xRz	P^*
{1,6}	7	$z\text{: } A \wedge B$	$\Box\,E$: 1,6
{1,6}	8	$z\text{:}B$	$\wedge\,E$: 7
{1}	9	$x\text{: } \quad B$	$\Box\,I$: 6,8
{1}	10	$x\text{: } \Box\,A \wedge \Box\,B$	$\wedge\,I$: 5,9

Example 2.2

{1}	1	$x: \Diamond (A \wedge B)$	P
{2}	2	$y: A \wedge B$	P^*
{3}	3	xRy	P^*
{2}	4	$y{:}A$	$\wedge E$: 2
{2,3}	5	$x: \Diamond A$	$\Diamond I$: 3,4
{1}	6	$x: \Diamond A$	$\Diamond E$: 1,2,3,5
{7}	7	$z: A \wedge B$	P^*
{8}	8	xRz	P^*
{7}	9	$z{:}B$	$\wedge E$: 7
{7,8}	10	$x: \Diamond B$	$\Diamond I$: 8,9
{1}	11	$x: \Diamond B$	$\Diamond E$: 1,7,8,10
{1}	12	$x: \Diamond A \wedge \Diamond B$	$\wedge I$: 6,11

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